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THE STANDARD IDENTITY IN CHARACTERISTIC p A CONJECTURE OF I.B. VOLICHENKO

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ABSTRACT

In this paper we present a full proof of the result which was announced at the International Congress of Mathematicians in Kyoto (August, 1990): Any associative PI-algebra over a field of characteristic p satisfies the standard identity.

THEOREM: *Every associative PI-algebra over a field of characteristic $p > 0$ satisfies the identity*

$$(1) \qquad \sum_{\sigma\in S(n)} x_{\sigma(1)}\cdots x_{\sigma(n)} = 0$$

for some n where ($S(n)$ is the symmetric group of order n).

In the sequel, we shall refer to (1) as the **symmetric identity** of degree n. This theorem implies a positive answer for I.B. Volichenko's conjecture posed in 1981:

COROLLARY: *Any (associative) PI-algebra over a field of characteristic $p > 0$ satisfies the standard identity of suitable degree.*

Indeed, let A be a PI-algebra over a field of characteristic $p > 0$; G the Grassman algebra over F generated by the elements $e_1, e_2, \ldots$, satisfying the relations

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$e_ie_j = e_je_i$ for all i,j. By Regev's theorem the algebra $A \otimes_F G$ is a PI-algebra; therefore, $A \otimes_F G$ will satisfy (1) for some n. Making in (1) the substitution

$$x_i = a_i \otimes e_i,$$

where $a_i \in A$, we get the equality

$$\left(\sum_{\sigma \in S(n)} (-1)^\sigma a_{\sigma(1)} \cdots a_{\sigma(n)}\right) \otimes e_1 \cdots e_n = 0,$$

which implies

$$\sum_{\sigma \in S(n)} (-1) a_{\sigma(1)} \cdots a_{\sigma(n)} = 0$$

for all $a_1, \ldots, a_n \in A$, i.e. the algebra A satisfies the standard identity of degree n.

§1. Preliminaries

Since the symmetric identities are multilinear, we may assume in the proof of the theorem that the basic field F is infinite with ch $F = p > 0$. Let B be the commutative algebra over F, generated by the elements $b_1, b_2, \ldots$ satisfying the relations

$$(2) \qquad b_i^2 = 0, \quad i = 1, 2, \ldots.$$

LEMMA 1: *An algebra A over an infinite field F satisfies the identity (1) if and only if the algebra $A \otimes B$ satisfies the identity $x^n = 0$.*

Proof: An arbitary element $x \in A \otimes_F B$ can be written in the form

$$x = \sum_i a_i \otimes c_i,$$

where $a_i \in A$, $c_i \in B$ and the elements c_i are the products of the b_j. (2) implies that $c_i^2 = 0$ for all i. Therefore x can be written as a linear combination of elements of the form

$$\sum_{\sigma \in S(n)} a_{i_{\sigma(1)}} \ldots a_{i_{\sigma(n)}} \otimes c$$

which equals 0, because the algebra A satisfies the identity (1).

The converse assertion is trivial because (1) is the full linearisation of the identity $x^n = 0$. ∎

Since B is a commutative non-nilpotent algebra, the multilinear identities of the algebras A, $A \otimes_F B$ are the same. Thus the main theorem of the paper asserts that any PI-algebra over a field of characteristic $p > 0$ satisfies the same multilinear identities as some nil-algebra of bounded index.

LEMMA 2: *If an algebra A satisfies the symmetric identity then an algebra $M_n(A)$ also satisfies the symmetric identity of suitable degree.*

Proof: By Lemma 1 it is sufficent to prove that the algebra $M_n(A \otimes B)$ satisfies the identity $x^N = 0$ for some N.

Let $x = \sum c_{ij}e_{ij} \in M_n(A \otimes B)$, where $c_{ij} \in A \otimes B$. By Lemma 1 $A \otimes B$ is a nil-algebra of a bounded index; therefore, the subalgebra C generated by the elements c_{ij} will be nilpotent. Let N_1 be the index of nilpotency of C; then $x^{N_1} = 0$. Thus taking $N \geq N_1$ to be the index of nilpotency of an n^2-generated free algebra of the variety $\mathrm{Var}(A \otimes B)$, the algebra $M_n(A \otimes B)$ satisfies the identity $x^N = 0$. ∎

LEMMA 3: *An associative algebra satisfying the Engel identity*

$$[x, y, \ldots, y] = 0 \tag{3}$$

of degree m satisfies the symmetric identity of suitable degree.

Proof: We may assume that $m = p^k$. It is easy to verify that for any $\lambda \in F$

$$(\lambda x + y)^m = y^m + \lambda[x, y, \ldots, y] + \lambda^2\varphi(x, y)$$

i.e. the identity (3) is a partial linearization of the identity $x^m = 0$. Therefore, by linearizing (3) we get an identity of the form (1). ∎

LEMMA 4: *Let $I \lhd A$. If I and A/I both satisfy symmetric identities, then A also satisfies a symmetric identity.*

Proof: By Lemma 1 the algebras $I \otimes B$, $(A/I) \otimes B$ satisfy the identities $x^{n_1} = 0$, $x^{n_2} = 0$ respectively. Thus $A \otimes B$ satisfies the identity $x^{n_1+n_2} = 0$. ∎

§2. The proof of the main theorem

Since as any PI-algebra can be embedded into a PI-algebra with unit, it is sufficient to prove the main theorem for PI-algebras with unit.

Let $F\langle x\rangle$ be the free associative algebra with unit generated by a countable set X, and let A be a PI-algebra with unit. Let F_n denote the n-generated relatively free algebra of the variety $\mathrm{Var}(A)$ with unit. In [1] it was proved that there exist finite-dimensional classical algebras C_n such that

$$\mathrm{Var}(F_n) = \mathrm{Var}(C_n). \tag{4}$$

(A finite-dimensional algebra is called **classical** if $\mathcal{C} = \mathcal{P}(C) + \mathrm{Rad}\ \mathcal{C}$, $\mathcal{P}(C) \cap \mathrm{Rad}\ \mathcal{C} = 0$, and $\mathcal{P}(C)$ is a direct product of full matrix algebras over the base field F.) We fix the finite-dimensional classical algebras satisfying (4).

Remark: If for some n the algebra C_n satisfies an identity $f(x_1, \dots, x_m) = 0$, where $m \le n$, then the algebra A also satisfies this identity.

Indeed, let Γ be the ideal of identities of the algebra A, Γ' the ideal of identities of the algebra F_n. Since $m \le n$ we have the inclusions

$$f(x_1, \dots, x_m) \in \Gamma' \cap F\langle x_1, \dots, x_n\rangle = \Gamma \cap F\langle x_1, \dots, x_n\rangle \subseteq \Gamma.$$

If $\mathcal{A}$ is a proper variety of associative algebras over F, then we denote by $\alpha(\mathcal{A})$ the maximal integer α satisfying the condition $M_\alpha(F) \in \mathcal{A}$. The number $\alpha(\mathcal{A})$ is called the **matrix complexity of the variety $\mathcal{A}$**.

A nontrivial identity of the form

$$\sum_{(i)} \alpha_{(i)} z^{i_1} y_1 z^{i_2} y_2 \cdots z^{i_m} y_m z^{i_{m+1}} = 0,$$

where $\alpha_{(i)} \in F$, is said to be an identity of algebraic form of order m. We denote by $\beta(\mathcal{A})$ the minimal order of an identity of the algebraic form is satisfied by all the algebras of $\mathcal{A}$. If some algebra of $\mathcal{A}$ does not satisfy any identity of algebraic form then we let $\beta(\mathcal{A}) = \infty$.

The ordered pair $(\alpha(\mathcal{A}), \beta(\mathcal{A}))$ is called the **type of the variety $\mathcal{A}$**. We order the types lexicographically: $(\alpha_1, \beta_1) \le (\alpha_2, \beta_2)$ if either $\alpha_1 < \alpha_2$ or $\alpha_1 = \alpha_2$, $\beta \le \beta_2$.

We shall prove the theorem by induction on the type $\mathrm{Var}(A)$.

1. BASE FOR THE INDUCTION. $\beta(\mathrm{Var}(A)) = 1$

In this case the algebra A satisfies a nontrivial identity of the form

$$\sum_{k=0}^{n} \alpha_k [y, z, \dots, z] z^{n-k}.$$

Hence, since A is an algebra with unit, we obtain (substituting $z+1$ for z and taking the nontrivial homogeneous component of minimal degree) that the algebra A satisfies an Engel identity. Then by Lemma 3 the algebra A satisfies a symmetric identity of suitable degree.

2. INDUCTIVE HYPOTHESIS.

Suppose that for every variety $\mathcal{A}$ of type less than the type of the variety $\mathrm{Var}(A)$ that all the algebras of $\mathcal{A}$ satisfy a symmetric identity of suitable degree.

3. Let $\alpha = \alpha(\mathrm{Var}(A))$, $\beta = \beta(\mathrm{Var}(A))$. We shall assume that $\beta > 1$.

The algebras C_n may be represented in a form

$$C_n = \mathcal{P}(C_n) + \mathrm{Rad}\, C_n,$$

where $\mathcal{P}(C_n)$ is the semisimple part of the algebra C_n, $\mathcal{P}(C_n) \cap \mathrm{Rad}\, C_n = 0$. We have the decomposition

$$\mathcal{P}(C_n) = C_n^{(0)} \oplus C_n^{(1)} \oplus \cdots \oplus C_n^{S(n)},$$

where $C_n^{(i)} \simeq M_\alpha(F)$ if $i > 0$, and the algebra $C_n^{(0)}$ is a direct sum of the full matrix algebras $M_j(F)$, $j < \alpha$.

Let $e_n^{(i)}$ be the unit of the algebra $C_n^{(i)}$. We denote by I_n the ideal of the algebra C_n generated by all the elements of the form $e_n^{(i)} c e_n^{(j)}$, where $j \neq i$, $c \in C_n$. It is obvious that $I_n \subseteq \mathrm{Rad}\, C_n$.

Now we will prove that if there exists a natural number q such that for each n the algebra I_n satisfies the symmetric identity of degree q, then the algebra A satisfies a symmetric identity of suitable degree.

We suppose first that $\alpha = 1$. Consider the algebra C_n, where $n = 4q$. The algebra C_n/I_n can be represented as a direct sum of local algebras, therefore it satisfies the identity

$$g(x, y, z, t) = z[x, \underbrace{y, \dots, y}_{m}]t = 0$$

for some m. By Engel's theorem C_n satisifes the identity

$$\sum_{\sigma\in S(q)} g_{\sigma(1)}\cdots g_{\sigma(q)} = 0 \tag{5}$$

where $g_i = g(x_i, y_i, z_i, t_i)$. Since the left side of the identity (5) depends on $4q = n$ variables, by the remark the algebra A satisfies the identity (5). It follows that there exist an ideal I of the algebra A such that the algebra A/I satisfies Engel's identity and the algebra I satisfies the symmetric identity of degree q. Applying Lemmas 3 and 4, we conclude that the algebra A satisfies a symmetric identity of suitable degree.

We suppose now that $\alpha > 1$. For each n the algebra C_n/I_n can be represented as a direct sum of the algebras of the form

$$C_n^{(i)} = e_n^{(i)} C e_n^{(i)} / I_n \cap e_n^{(i)} C_n e_n^{(i)}.$$

If $i > 0$ then we have the isomorphism

$$C_n^{(i)} \simeq M_\alpha(\mathcal{D}_n^{(i)}),$$

where $\mathcal{D}_n^{(i)}$ are suitable algebras. Consider

$$C = M_\alpha(F\langle x\rangle)/\Gamma(M_\alpha(F\langle x\rangle)),$$

where Γ is the ideal of identities of the algebra A, and $\Gamma(M_\alpha(F\langle x\rangle))$ is the ideal of $M_\alpha(F\langle x\rangle)$ generated by all the elements of a form $f(a_1,\dots,a_m)$ where $f(x_1,\dots,x_m) \in \Gamma$, $a_i \in M_\alpha(F\langle x\rangle)$. It is obvious that for all n and $i > 0$ the algebra $C_n^{(i)}$ satisfies all the identities of the algebra C.

We have $C = M_\alpha(\mathcal{D})$ for some algebra $\mathcal{D}$. Since C is a PI-algebra then $\mathcal{D}$ is also a PI-algebra. It is easy to see that $\alpha(\mathrm{Var}(\mathcal{D})) = 1 < \alpha$. Thus by the inductive hypothesis the algebra $\mathcal{D}$ satisfies a symmetric identity of suitable degree. Then by Lemma 2 the algebra C also satisfies a symmetric identity of suitable degree m.

So we have proved that there exists a natural number m such that for any n and any $i > 0$ the algebra $C_n^{(i)}$ satisfies a symmetric identity of degree m.

We fix an arbitrary polynomial $u(x_1,\dots,x_r) \in F\langle x\rangle$ satisfying the following properties:

1. $u = 0$ is an identity of the algebras $M_j(F)$ for $j < \alpha$;

2. for each k the algebra $M_\alpha(F)$ does not satisfy the identity $u^k = 0$.

Let $n = (m + r + 3)q$. Since the algebra $M_j(F)$ for $j < 2$ satisfies the identity $u = 0$, then for some k the algebra $C_n^{(0)}$ satisfies the identity $u^k = 0$. This and what was proved above imply that the algebra C_n/I_n satisfies the identity

(6)
$$g(z, t, v, x_1, \ldots, x_m, y_1, \ldots, y_r) = z \sum_{\sigma \in S(m)} x_{\sigma(1)} \cdots x_{\sigma(m)} t(u(y_1, \ldots, y_r)^k)v = 0.$$

Then the algebra C_n satisfies the identity

$$\sum_{\sigma \in S(g)} g_{\sigma(1)} \cdots g_{\sigma(m)} = 0 \tag{7}$$

where

$$g_i = g_i(z^{(i)}, t^{(i)}, v^{(i)}, x_1^{(i)}, \ldots, x_m^{(i)}, y_1^{(i)}, \ldots, y_r^{(i)}).$$

Since the left side of the identity (7) depends on $(m+r+3)q = n$ variables, then by the remark the algebra A also satisfies the identity (7). It follows that there exists an ideal I of the algebra A such that the algebra I satisfies a symmetric identity of degree q and the algebras A/I satisfies the identity (6).

Let $\mathcal{A}$ be the variety of algebras satisfying the identity $u^k = 0$. By construction of the polynomial u we have the inequality $\alpha(\mathcal{A}) < \alpha$. Thus by the inductive hypothesis the algebras of the variety $\mathcal{A}$ satisfy a symmetric identity of suitable degree, i.e. the identity $u^k = 0$ implies a symmetric identity of suitable degree. Hence the identity (6) also implies a symmetric identity. Thus we have proved that the algebra A/I satisfies a symmetric identity. Finally, applying Lemma 4 we conclude that the algebra A satisfies a symmetric identity.

4. We fix an arbitrary central polynomial $h(x_1, \ldots, x_r)$ of the algebra M_α. Let $n = r + 2$. Since the algebra $C_n/\mathrm{Rad}\ C_n$ satisfies the identity $[h, y] = 0$ and the ideal Rad C_n is nilpotent, the algebra C_n satisfies for some k every identity of the form

$$z^{m_r}[h(x_1, \ldots, x_r), y]z^{m_r} \cdots z^{m_{2k}}[h(x_1, \ldots, x_r), y]z^{m_{2k+1}} = 0 \tag{8}$$

where $m_i \geq 0$. The left side of this identity depends on $r + 2 = n$ variables; therefore, by the remark the identity (8) holds in the algebra A and in the algebras C_n for all n.

In this part of the proof we get some generalized identities of C_n which will be needed later.

Suppose that the set of units $\{e_n^{(i)},\ i \geq 1\}$ contains $2k+1$ distinct elements $e_1, \ldots, e_{2k+1}$. Since $h(x_1, \ldots, x_r)$ is a central polynomial of $M_\alpha(F)$, for all $\lambda_1, \ldots, \lambda_{2k+1} \in F$ we can choose the elements $a_1, \ldots, a_r \in \mathcal{P}(C_n)$ such that

$$h(a_1, \ldots, a_r) = \sum_{i=1}^{2k+1} \lambda_i e_i.$$

Make the following substitution in the identity (8)

$$x_i = a_i; \quad y = \sum_{i=1}^{2k} e_i y_i e_{i+1}, \quad y_i \in C_n; \quad z = 1.$$

We obtain the equality

$$\prod_{i=1}^{2k} (\lambda_i - \lambda_{i+1}) e_1 y_1 e_2 y_2 \cdots e_{2k} y_{2k} e_{2k+1} = 0.$$

Since F is an infinite field, we can choose the elements λ_i such that $\lambda_i \neq \lambda_j$ if $i \neq j$. Thus for all distinct units $e_1, \ldots, e_{2k+1} \in \{e_n^{(i)} \mid i \geq 1\}$ the algebra C_n satisfies the generalized identity

$$e_1 y_1 e_2 y_2 \cdots e_{2k} y_{2k} e_{2k+1} = 0. \tag{9}$$

Let e_1 be an arbitrary unit belonging to the set $\{e_n^{(i)} \mid i \geq 0\}$, $e_2 = 1 - e_1$. If $e_1 \neq e_n^{(0)}$ then we can choose the elements $a_i \in \mathcal{P}(n)$ such that $h(a_1, \ldots, a_r) = e_1$. If $e_1 = e_n^{(0)}$ the elements a_i can be chosen so that $h(a_1, \ldots, a_r) = e_2$. Make the substitution into the identity (8): $x_i = a_i,\ y = e_1 y e_2,\ z = z$, where $y, z \in C_n$. We obtain the generalized identity of the algebra C_n,

$$z^{m_1} e_1 y e_2 \cdots z^{m_{2k}} e_1 y e_2 z^{m_{2k+1}} = 0 \tag{10}$$

for all $m_i \geq 0$.

5. Now we prove the following statement:

If there exists a natural number N such that for any n and i the algebra $e_n^{(i)} I_n e_n^{(i)}$ satisfies a symmetric identity of degree N, then for all n the algebra I_n satisfies the symmetric identity

$$\sum_{\sigma \in S(q)} y_{\sigma(1)} \cdots y_{\sigma(q)} = 0 \tag{11}$$

where

$$q = ((2k)^2(2k+1)+1)N+1.$$

Indeed, since the identity (11) is multilinear then it is sufficient to verify the identity for the elements

$$y_i = e_{k_i} y_i e_{t_j},$$

where $e_{k_i}, e_{t_j} \in \{e_n^{(i)} \mid i \geq 0\}$, $e_j \neq e_{j'}$ if $j \neq j'$, $y_i \in I_n$.

It follows from the identity (9) that if the set $\pi = \{e_{k_1}, e_{t_1}, \ldots, e_{k_q}, e_{t_q}\}$ contains more than $2k+1$ elements (i.e., if $|\pi \cap \{e_n^{(i)} \mid i \geq 1\}| \geq 2k+1$) then the left side of the identity (11) equals zero for the substitution we have made.

An element y_i is called **mixed** if $k_i \neq t_i$. If the number of a mixed element is greater than $(2k)^2(2k+1)$ then among the elements y_i there exist $(2k+1)$ mixed elements $y_{i_1}, \ldots, y_{i_{2k+1}}$ such that $k_{i_y} = k_{i_1}$, $t_{i_j} = t_{i_1}$ for all $j = 1, 2, \ldots, 2k+1$. Then the left side of (11), for the substitution we have made, can be represented as a linear combination of expressions which are the linearizations of the left sides of the identities of the form (10), where $e_1 = e_{k_{i_1}}$, $e_2 = 1 - e_1$, i.e. the left side of (11) again equals zero.

So we may assume that the number of the mixed elements $\leq (2k)^2(2k+1)$. In the case the left side of the identity (12) for the substitution we have made can be represented as a linear combination of the expressions of the form

$$u \sum_{\sigma \in S(N)} y_{i_{\sigma(1)}} \cdots y_{i_{\sigma(N)}} V, \tag{12}$$

where y_{i_j} are nonmixed elements. The expression (12) equals zero because the nonmixed elements belong to the subalgebras $\bigoplus_{i \geq 0} e_n^{(i)} I_n e_n^{(i)}$ which satisfies the symmetric identity of degree N. The statement is proved.

Let us continue the proof of the main theorem. Consider the variety A generated by all the algebras $e_n^{(i)} I_n e_n^{(i)}$, where $n = 1, 2, \ldots$, $i = 1, 2, \ldots, s(n)$. It remains to prove that all the algebras belonging to $\mathcal{A}$ satisfy a symmetric identity of suitable degree.

Since $\mathcal{A} \subseteq \operatorname{Var} A$ then $\alpha(\mathcal{A}) \leq \alpha = \alpha(\operatorname{Var}(A))$, $\beta(\mathcal{A}) \leq \beta = \beta(\operatorname{Var}(A))$; therefore, by the inductive hypothesis it is sufficient to prove that $\beta(\mathcal{A}) < \beta$.

6. We suppose first that $\beta \neq \infty$. Then the algebra A satisfies a nontrivial identity of the form

$$\sum_{(i)=(i_1,\ldots,i_\beta)} \alpha_{(i)} z^{i_1} y_1 z^{i_2} y_2 \cdots z^{i_{\beta-1}} y_{\beta-1} z^{i_\beta} y_\beta z^{i_{\beta+1}} = 0. \tag{13}$$

We show that for all u, i the algebra $e_n^{(i)} I_n e_n^{(i)}$ satisfies the identity $d_m = 0$, where

$$g_m(z, y_1, \ldots, y_{\beta-1}) = \sum_{\substack{(i) \\ i_\beta = m}} \alpha_{(i)} z^{i_1} y_1 \cdots z^{i_{\beta-1}} y_{\beta-1} z^{i_\beta+1}.$$

Indeed, let $e_1 = e_n^{(i)}$. It follows from the definition of the ideal I_n that

$$e_1 I_n e_1 = e_1 C_n e_2 C_n e_1, \text{ where } e_2 = 1 - e_1.$$

Make the following substitution into the identity (13):

$z = \lambda e_2 + d$, where $d \in e_1 I_n e_1$, $\lambda \in F$;

$y_i = d_i$, where $d_i \in e_1 I_n e_1$, $i = 1, \ldots, \beta - 2$

$y_{\beta_1} = c_1$, $y_\beta = c_2$, where $c_1 \in e_1 C_n e_2$, $c_2 \in e_2 C_n e_1$.

We get the result:

$$\sum_m \lambda^m g_m(d, d_1, \ldots, d_{\beta-2}, C_1 C_2) = 0.$$

Since F is an infinite field, the equality

$$g_m(d, d_1, \ldots, d_{\beta_2} C_1 C_2) = 0$$

follows for all m.

An arbitrary element $d_{\beta-1} \in e_1 I_n e_1$ can be represented as a linear combination of the elements of the form cc', where $c \in e_1 c_n e_2$, $c' \in e_2 C_n e_1$. Therefore we get the equality

$$g_m(d, d_1, \ldots, d_{\beta-1}) = 0$$

for all $d, d_i \in e_1 C_n e_1$, i.e. the algebra $e_1 I_n e_1$ satisfies the identity $g_m = 0$.

Since the identity (13) is nontrivial, then for some m the identity $g_m = 0$ is a nontrivial identity of algebraic form of the order $\beta - 1$. Thus $\beta(\mathcal{A}) < \beta$.

7. We suppose finally that $\beta = \infty$. We prove in that case that every algebra belonging to the variety $\mathcal{A}$ satisfies some identity of the algebraic form of order k (k is the same number as in the formula (8)), i.e. $\beta(\mathcal{A}) \leq k < \beta$.

Let $e_1 = e_n^{(i)}$, $e_2 = 1 - e$. Choose elements $a_1, \ldots, a_r \in C_n$ such that either $h(a_1, \ldots, a_r) = e_1$ or $h(a_1, \ldots, a_r) = e_2$ (cf. item 4). We make the following substitution into the identity (8) for $m_2 = m_4 = \cdots = m_{2k} = 0$, $m_{2i-1} = n_i$: $x_i = a_i$, $y = \overline{y} + \tilde{y}$, where $\overline{y} \in e_1 C_n e_2$, $\tilde{y} \in e_2 C_n e_1$, $z \in e_1 C_n e_1$. If $m_1 = 0$

we multiply the result by e_1 from the left. As a result we get the following generalized identity of C_n :

$$z^{n_1}\overline{y}\tilde{y}z^{n_2}\overline{y}\tilde{y}z^{n_3}\cdots\overline{y}\tilde{y}z^{n_{k+1}}=0 \tag{14}$$

for every integer $n_i \geq 0$ and any $\overline{y} \in e_1C_ne_2$, $\tilde{y} \in e_2C_ne_1$, $z \in e_1C_ne_2$.

Now we shall describe a process of transforming of generalized identities which is similar to the process of linearization.

Suppose the polynomial $f = f(y, z, x_1, \ldots, x_s)$ is homogeneous with respect to y, $\deg_y f = \mu$ (the variable z may appear in the polynomial f vacuously). Then f can be written in the form

$$f = \sum_{(u)} \alpha_{(u)}u_1yu_2y\cdots u_\mu yu_{\mu+1})$$

where u_i are the words on the variables $x_1, \ldots, x_s$, and $\alpha_{(u)} \in F$. For any vector $\vec{n} = (n_1, \ldots, n_\mu)$ of non-negative integers, we define the polynomials

$$\ell(\vec{n}) = \sum_{(u)} \alpha_{(u)}u_1z^{n_1}yu_2z^{n_2}y\ldots z^{n_\mu}yu_{\mu+1},$$

$$r(\vec{n}) = \sum_{(u)} \alpha_{(u)}u_1yz^{n_1}u_2yz^{n_2}\ldots yz^{n_\mu}yu_{\mu+1}.$$

Consider an associative algebra $\mathcal{D}$ over the field F which contains fixed subsets $M_y, M_z, M_{x_1}, \ldots, M_{x_s}$ satisfying the properties:

1. M_y is a linear F subspace.
2. Either $M_zM_y \subseteq M_y$, or $M_yM_z \subseteq M_y$.

We assume that for any vector $\vec{n} = (n_1, \ldots, n_\mu)$ and for any elements $y \in M_y$, $z \in M_z$, $x_i \in M_{x_i}$ the following equality holds in $\mathcal{D}$: $g(\vec{n}) = 0$, where $g(\vec{n}) = \ell(\vec{n})$ if $M_zM_y \subseteq M_y$, $g(\vec{n}) = r(\vec{n})$ if $M_yM_z \subseteq M_y$.

Since M_y is a linear space over an infinite field F then for any $y_1, y \in M_y$, $z \in M_z$, $x_i \in M_{x_i}$ and for any integer vector $\vec{n}$ the following equality holds in $\mathcal{D}$;

$$\sum_{i=1}^{\mu} g_i(\vec{n}) = 0, \tag{15}$$

where

$$g_i(\vec{n}) = \sum_{(u)} \alpha_{(u)}u_1z^{n_1}y\cdots u_{i-1}z^{n_{i-1}}yu_iz^{u_i}y_1u_{i+1}z^{n_{i+1}}y\cdots u_\mu z^{n_\mu}yu_{\mu+1})$$

if $g(\vec{n}) = \ell(\vec{n})$,

$$g_i(\vec{n}) = \sum_{(u)} \alpha_{(u)} u_1 y z^{n_1} \cdots u_{i-1} y z^{n_{i-1}} u_i z^{u_i} y_1 z^{n_{i+1}} u_{i+1} y z^{n_{i+1}} t \cdots u_\mu y z^{n_\mu} u_{\mu+1})$$

if $g(\vec{n}) = z(\vec{n})$. This equality is a partial linearization of $g(\vec{n})$ with respect to y.

We denote by ε_i the vector

$$\begin{pmatrix} 0, \dots, 0, \underset{i}{1}, 0 \dots 0 \end{pmatrix}$$

We substitute into the equality (15) $y_1 = z^j y_1$ if $M_z M_y \subseteq M_y$, $y_1 = y_1 z^j$ if $M_y M_z \subseteq M_y$. We get the result

$$\sum_{i=1}^{\mu} g_i(\vec{n} + j\vec{\varepsilon}_i) = 0, \tag{16}$$

which is true for all $\vec{n}, j$.

Consider the polynomial

$$h(\vec{n}) = \sum_{\nu=0}^{\mu-1} (-1)^\nu \sum_{z \le j_1 < \cdots < j_\nu \le \mu} \sum_{i=1}^{\mu} g_i(\vec{n} + \vec{\varepsilon}_{j_1} + \cdots + \vec{\varepsilon}_{j_\nu} + (\mu - \nu)\vec{\varepsilon}_i). \tag{17}$$

This polynomial is a linear combination of polynomials of type (16). Thus the equality $h(\vec{n}) = 0$ holds in $\mathcal{D}$ for every $\vec{n}$.

Changing the order of summation in (17) we obtain

$$h(\vec{n}) = \sum_{i=1}^{u} h_i(\vec{n}),$$

where

$$h_i(\vec{n}) = \sum_{\nu=0}^{\mu-1} (-1)^\nu \sum_{z \le j_1 < \cdots < j_\nu \le \mu} g_i(\vec{n} + \vec{\varepsilon}_{j_1} + \cdots + \vec{\varepsilon}_{j_\nu} + (\mu - \nu)\vec{\varepsilon}_i).$$

It is easy to see that $h_i(\vec{n})$ is trivial for $i \ge 2$. It is obvious also that if the initial polynomial f is nontrivial then the polynomial $h_1(\vec{n})$ is nontrivial also.

We let

$$\mathcal{L}_{y, y_1}(f) = h(\vec{0}).$$

Now we apply the described process for the case $\mathcal{D} = C_n$, $M_{\overline{y}} = e_1 C_n e_2$, $M_{\tilde{y}} = e_2 C_n e_1$, $M_z = e_1 C_n e_1$, $f = (\overline{y}\tilde{y})^k$. By what was proved above for all $\overline{y}, \overline{y}_i \in M_{\overline{y}}$, $\tilde{y}, \tilde{y}_i \in M_{\tilde{y}}$, $z \in M_z$, the following equality holds in C_n.

$$\mathcal{L}_{\tilde{y},\tilde{y}_k}(\mathcal{L}_{\tilde{y},\tilde{y}_{k-1}}(\ldots \mathcal{L}_{\tilde{y},\tilde{y}_1}(\mathcal{L}_{\overline{y},\overline{y}_k}(\ldots(\mathcal{L}_{\overline{y},\overline{y}_1}(f)\ldots) = 0.$$

The left-hand side of this equality is a nontrivial polynomial of the form

$$\sum_{(i)} \beta_{(i)} z^{i_1} \overline{y}_1 \tilde{y}_1 z^{i_2} \overline{y}_2 \tilde{y}_2 \cdots z^{i_k} \overline{y}_k \tilde{y}_k z^{i_{k+1}}.$$

In particular, this polynomial is linear with respect to the variables $\overline{y}_i, \tilde{y}_i$. Since $M_{\overline{y}} M_{\tilde{y}} = e_1 I_n e_1$ it follows that the algebra $e_1 I_n e_1$ satisfies the identity

$$\sum_{(i)} \beta_{(i)} z^{i_1} y_1 z^{i_2} y_2 \cdots z^{i_i} y_k z^{i_{k+1}} = 0 \tag{18}$$

which is a nontrivial identity of algebraic form of order k. Since the coefficients $\beta_{(i)}$ do not depend on n and e_1 then all the algebras of the variety $\mathcal{A}$ satisfy the identity (18). The Theorem is proved.

References

[1] A.R. Kemer, *Identities of the finitely-generated algebras over infinite field*, Izvestia Academii Nauk SSSR Seria Mat. **54, No. 4** (1990), 726–753 (in Russian; translated in Math. Isvestia).